Category Theory

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An short introduction to Category Theory. Version 0.6

1 Introduction

When I studied theoretical physics in the seventies nobody in my environment talked about catergory theory although categories were a long time before introduced by Samuel Eilenberg and Saunders Mac Lane in 1942-45 in connection with algebraic topology. Today, if you read papers on mathematics, many assume some knowledge of category theory. Quite often you will see some terminology from category theory used in books on mathematics but also in books on type theory, languages and data structures in computer science.

This paper is a very short introduction to the terminology. I am no expert and I hope there are not many errors. I start with a rehearsal of the usual subjects of mathematics to show that many of those subjects have some structure together with morphisms (functions, binary operations, cuntinuous functions or wathever) that work on those structures. Those morphisms are the central objects in category theory, where the subject is what can be said about the morphisms without considering the kinds of structures they work on. This rehearsal can be skipped of course but it is written to be self-contained when giving illustrations of the concepts of category theory.

After that the introduction on category theory follows. Finally some links to internet sites are mentioned. Especially the short lectures on YouTube by the Catsters are a pleasure to watch and many give the same information as this introduction.

2 Some mathematical structures

2.1 Sets

A set has elements. Every element is in some way identified, every element of the set is different. Two sets are equal when both sets have the same elements. Basic operations on two sets A, B are the union $A \cup B$, the intersection $A \cap B$ and the difference $A \setminus B$ (the set $\{x : x \in A, x \notin B\}$). The (Carthesian) product of the two sets, $A \times B$, is the set $\{(a, b) : a \in A, b \in B\}$. The disjoint union ofthe two sets, $A \coprod B$, is the set $\{(a, i_A) : a \in A\} \cup \{(b, i_B) : b \in B\}$. The major difference with the normal union is that in a disjoint union it is known that the elements come from a particular set, A or B, when they intersect. The i_A and i_B act as labels to differentiate between the elements from both sets.

A (total) function $f : A \to B$ maps all the elements from one set A on another set B. A function is seen as a subset of $A \times B$. It is allowed that the set B is the same set as A (a function

$A \to A$). And it is allowed that the set A is empty (the empty function: $\emptyset \to B$; for every set B there is only one such function, because there is only one subset of $\emptyset \times B$, which is the $\emptyset$ itself!). A function is surjective when all elements of B have a corresponding origin in A (the function is 'onto'). A function is injective when different elements in A are mapped to different values in B. The domain of the function is A. The codomain of the function is B. The range of the function is the set of elements in B that have an origin in A. When the function is not surjective then the range is a proper subset of the codomain. A function that is injective and surjective is a bijection. A bijection has an inverse because every element of B has an origin in A (because the funtion is surjective) and it has only one origin in A (because the function is injective). This means that to every element of A correponds exactly one element of B. Both sets have the same number of elements and the isomorphism can be seen as a re-labeling of the elements. An example is the set of odd numbers and the set of even numbers where the isomorphism is the relabeling of an odd number i with $i + 1$.

A partition of a set A is a division of A into non-overlapping subsets, called equivalence classes, that together form a cover of A. An equivalence relation on a set gives a partitioning of a set. An equivalence relation is a relation $\equiv$ such that for all elements a, b, c of the set A the following holds:

1. reflexivity: $a \equiv a$
2. transitivity: if $a \equiv b$ and $b \equiv c$ then $a \equiv c$
3. symmetry: if $a \equiv b$ then $b \equiv a$

Notation: all elements that have an equivalence relation with the element a form the equivalence set $[a]$. The intersection of two equivalence relations is again an equivalence relation. This makes it possible to speak of the equivalence relation generated by any relation R as the smallest equivalence relation containing R. The smallest equivalence relation on a set is the one where every element has only a relation with itself, but of course that relation does in general not have R as a subset. The largest equivalence relation is the whole set $A \times A$: the partition gives one equivalence class containing al the elements of A.

2.1.1 Examples of functions

- for each set A there is an identity mapping $id_A : A \to A$ that maps an element to the same element
- for each set $S \subset A$ there exists the inclusion map $i : S \to A$ that sends each element of S to the same element but now as an element of A. For the inclusion map the symbol $\hookrightarrow$ is used.
- as mentioned before, if a function $f : A \to B$ is a bijection there exists an inverse function $g : B \to A$. The composition $gf : A \to A$ is equal to id_A and the composition $fg : B \to B$ is equal to id_B. A function with such an inverse is called an isomorphism and for sets it is true that a bijection is an isomorphism.
- an example of a function that is not surjective is the function $f : \{1, 2\} \to \{1, 2, 3, 4, 5, 6\}$ with $f(1) = 3$ and $f(2) = 4$. Here there exists a function $g : \{1, 2, 3, 4, 5, 6\} \to \{1, 2\}$ with $g(i) = 1$ for i an odd number and $g(i) = 2$ for i an even number. Then $gf = id_A$, but $fg \neq id_B$.

2.1.2 Sets, small sets and classes

The powerset of a set A has as members all the possible subsets of A and is written as $\mathcal{P}(A)$. If A is finite and has as members $a_0, a_1, ..., a_n$ then of course $\{a_0\}$, $\{a_1\}$,... $\{a_n\}$ among others, are members of the powerset and from this it is evident that the powerset has more members than A itself. But this statement is also true when A is not finite: there is never a one-to-one correspondence between the members of A and the members of $\mathcal{P}(A)$. This was proved by Cantor, who also introduced the notion of equivalence of sets when a one-to-one correspondence between the members of two sets can be defined (giving rise for instance to the equivalence of the sets $\{n \in \mathbb{N}\}$ and $\{n^2 : n \in \mathbb{N}\}$) . Using Cantor's theorem on power sets Russel found paradoxes in the foundations of set theory. To avoid these paradoxes of set theory the concept of a class is introduced as something different from a set. Every set is a class. A class that is not a set is called a proper class, and a class that is a set is sometimes called a small class.

2.2 Ordered sets

A partially ordered set (poset) is a set together with a partial order between the elements of the set. A partial order is a relation $\leq$ such that for all elements a, b, c of the set A the following holds:

1. reflexivity: $a \leq a$
2. transitivity: if $a \leq b$ and $b \leq c$ then $a \leq c$
3. antisymmetry: if $a \leq b$ and $b \leq a$ then $a = a$

In a total order one has the extra totallity condition that $a \leq b$ or $b \leq a$, in which case one can forget the reflexivity because that becomes a direct consequence of the totality and antisymmetry. And finally a well-ordered set has a total order with the extra property that every non-empty subset of the set A has a least element.

2.3 Monoids

A monoid is a set, M, together with a binary 'product' $\cdot$ that forms from any two elements a and b another element in M, $a \cdot b$. The binary operation must satisfy the following requirements:

1. it is associative: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
2. there is one element in G, the identity e, with for all elements g of G $e \cdot g = g = g \cdot e$

In general the product is not commutative ($a \cdot b \neq b \cdot a$). A monoid can be seen as a set together with two functions $f_l a, f_r b : M \to M$ for each element a that sends each element b to $a \cdot b$ resp. to $b \cdot a$ (left and right multiplication with a). Both sets of functions are not independent of course. Once all functions $f_l a$ are known, so are the $f_r a$.

2.3.1 Examples of monoids

- the set of all finite strings over some fixed alphabet A forms a monoid with string concatenation as product. The empty string serves as the identity element. This monoid is denoted $A*$ and is called the free monoid over A.

2.4 Groups

A group is a set, G, together with a binary 'product' $\cdot$ that forms from any two elements a and b another element in G, $a\cdot b$. The binary operation must satisfy the following requirements:

1. it is associative: $(a\cdot b)\cdot c = a\cdot (b\cdot c)$
2. there is one element in G, the identity e, with for all elements g of G $e\cdot g = g = g\cdot e$
3. for all elements g in G there is a unique inverse g^{-1} in G, so that $g\cdot g^{-1} = e = g^{-1}\cdot g$

In general the product is not commutative. If it is commutative the group is called an abelian group (for all $g1, g2$ in G : $g1\cdot g2 = g2\cdot g1$). A generating set S of a group G is a subset such that every element of the group can be expressed as the product of finitely many elements of the subset and their inverses. Let G together with a binary product $\cdot$ and H together with a binary product $*$ be groups. A group homomorphism is a mapping f:$G \to H$ such that for every a and b: $f(a\cdot b) = f(a) * f(b)$.

A representation of a group G is a homomorfism from the group to another group H consisting of automorphisms h: $H \to H$. The most important example is a representation when H is a linear vectorspace.

A (left) group action of a group G on a set X is a binary operator $\circ : G \times X \to X$ that satisfies the conditions of associativity, $g1 \circ (g2 \circ x) = (g1\cdot g2) \circ x$ and of identity $e \circ x = x$. The group G is said to act on X (on the left). Every g is mapped by $\circ$ to a mapping $X \to X$ and gh is mapped to the composition $g(h(x))$. So, a group action can also be defined as a homomorphism from G to the symmetric group $SYM(X)$ of all bijecties $X \to X$.

2.4.1 Examples of groups

- the trivial group. A trivial group has one element, the identity. The inverse of the identity is the identity itself: $e\cdot e = e$
- the free group generated by a set S. The group of integers with + as 'group product' is free. We can take $S = \{1\}$.
- let G and H be groups. Then the free product of G and H is the most general group that can be made from generating sets of G and H. This group has G and H as subgroups.
- operations of symmetry. Some figures can be transformed to the same figure doing a rotation aroun e.g. 120 degrees. Define the group G having an element g1 that represents this rotation. And let e be the identity element representing no rotation. g1.g1 is another element of this group and g1.g1.g1 represents a rotation around 360 degrees, so = e. All the elements have an inverse in the group. This is not a free group because there is the extra condition for the generator of the group tha g1.g1.g1=e.
- for every group G another group called the opposite group G^{op} can be defined as the group having the same elements but a product $\cdot^{op}$ defined as ... for every element g1, g2 of the group. Of course for abelian groups the opposite group and the group itself are equal.
- the symmetric group S_N of N elements is the group whose elements are all the permutations of the N elements and has composition of permutations as the group product. There are N! elements in the group. The group acts on the set $\{1, 2, ...N\}$.

- if G1 and G2 are groups then the direct product G of these groups is a new group with as members the ordered pair $\{(g1, g2) : g1 \in G1, g2 \in G2\}$ and as group product $(g1, g2) * (g1', g2') = ((g1.g1'), (g2.g2'))$

2.5 Graphs

A graph $G(V, E)$ is a set of vertices V (also called nodes) and a set of edges E, where an edge connects two vertices. Let x and y be two nodes in the graph G. If you can reach y starting from x and following edges in the graph, then you say there is a path between x and y.

There are all kinds of graphs. Some are directed: the edges connect node one with node two, but not node two with node one. Some allow for loops: an edge comes from node one and goes back to node one. Some allow for multiple edges between the same two nodes. For disconnected graphs it is not true that all vertices are connected by a path.

Let G and H be graphs. A graph homomorfism is a mapping f:$G \rightarrow H$ such that is xy is an edge in G (from node x to node y, then $f(x)f(y)$ is an edge in H. So, whenever there is an edge in G the image is an edge also. It is not necessary that the number of nodes in H is equal or larger than in G. It is for instance possible to have a homeomorphism from a pentagon to a triangle. If the labels of the five nodes of the pentagon are A, B, C, D an E and the labels of the three nodes of the triangle are 1, 2 and 3 then the mapping A→0, B→2, C→3, D→1 and E→2 is a homomorphism. If it is not allowed that different nodes are mapped to the same target one talkes about injective homomorphisms.

2.5.1 Examples of graphs

- a complete graph K_N is defined as the graph with N vertices and with all pairs of vertices connected by an edge. Such a graph can be used to define the vertex coloring problem. In the vertex coloring problem you try to assing colors to the vertices such that an edge has always two different colors at its ends. It is understood that the graphs have no loops (the vertices connected by a loop must by definition be the same). To color a graph G with N different colors is the same problem as finding a homeomorfism from the graph to a complete graph K_N (having different colors for all its vertices). In the example of the homomorfism from the pentagon to the triangle there exists a 3-coloring of the vertices because there is a homeomorphism from the pentagon to the triangle (=K_3). Instead of using the labels 1, 2 and 3 use three different colors as labels, and you know how to color the vertices of the pentagon.

- the intersections of sets can be represented by an intersection graph with a node for each set and an edge between nodes when the corresponding sets do intersect. An example is the line graph of a given graph where a set is formed for each edge and where the vertices that are joined by the edge are the members of each set.

If a axiomatic approach to a subject is developed first a set of primitive notions is listed. For instance when Hilbert is introducing his approach to (Euclids) geometry he introduces five primitive notions: point, line, plane, congruence and 'inbetweenness' (points on a line between two other points). After that the axioms of the theory must be presented: statements in terms of the primitive notions that are the basic theorems of the theory. And then first order logic is used to deduce other theorems. In these deductions nothing may be assumed about the primitive notions except what is stated in the axioms.

2.6 Topology

The idea behind topology is to be able to define continuous functions from a set T to a set U. This is done by defining environments around all points of a set. Loosely speaking: a function is continuous in a point t of the set T when in taking smaller and smaller environments of that point t the environment of the targets of the function will get smaller and smaller too. Environments of a point are defined using open sets.

First we define a topology τ on a set T. The members of τ are called open sets just mentioned. In T. a family of subsets τ of T is a toplology on T if:

1. the empty set $\emptyset$ and T are elements of τ
2. any union of elements of τ is element of t
3. any intersection of finitely many elements of τ is an element of τ

A subset of T is closed when its complement in T is open. A set with a topology τ is caled a toplogical space.

An environment of a point t is a subset V of T, such that there is an open set that is subset of V and also has t as an element of the open set. A topological space is called a Haussdorff space (or a T2 space) if for all different elements there exists a disjoint environment. Almost all spaces are Hausdorff spaces.

A function f: $T \to U$ is continuous at some point t of T if and only if for any neighborhood V_U of $f(t)$, there is a neighborhood V_T of t such that $f(V_T) \subset V_U$. Another concept is continuity of a function (not continuity at a certain point). A function $T \to U$ is continuous if for every open set O in U the pre-image $f^{-1}(O)$ is open in T. A map is continuous if and only if it is continuous at every point. Related to continuity is of course a sequence converging to a point. A sequence $\{x_i\}_{i\in\mathbb{N}}$ converges to $x \in T$ if for every environment of x there is a $N \in \mathbb{N}$ such that all $\{x_i\}$ in that environment for $i > N$.

A bijective continuous function with continuous inverse function is called a homeomorphism. If there exists a homeomorphism f: $T \to U$ then T and U are said to be homeomorphic. They are the same 'up to a homeomorphism'. A topologist studies properties of spaces 'up to a homeomorphism'. A doughnut and a coffeecup are the 'same' to him since there exists a homeomorfism between the two objects. A property that is the same for any two homeomorphic spaces is a toplogical invariant. In general to prove that two spaces are homeomorphic one has to find a homeomorphism and to prove that two spaces are not homeomorphic one has to find an invariant property that distinguishes the two.

A toplogical space is said to be connected if it can not be represented as the union of two disconnected open sets. A path in T is a continuous map from $[0, 1]$ to T. A topological space is path connected when for every two points a and b there is a path from a to b. An arc in T is an injective continuous map from $[0, 1]$ to T. Any path connected space is connected (the inverse statement is not true). A topological space is arc connected when for every two points a and b there is an arc from a to b. For Haussdorf spaces the notions of path connected and arc connected spaces coincide.

2.6.1 Examples

- a topology on the real numbers $\mathbb{R}$ defined by al the open subsets (a,b). If we have a function f: $\mathbb{R} \to \mathbb{R}$ that is discontinuous in a point t then every map of a open set containing this point will contain the discontinuity.

- a topology on T defined by the following open sets: the empty set $\emptyset$, T itself and every subset of T. In particular every set $\{t\}$ for each point t that is element of T is an open set. This is the discrete topology on T. Every function is continuous in this topology.

- a topology on T defined by the following open sets: the empty set $\emptyset$ and T itself are the only open sets. The only continuous functions are constant functions.

- a topology on T defined by the open sets of a metric function $d(x, y)$: the 'balls' $d(x, y) < r$. A metrix function is a positive symmetric function $T \times T \to \mathbb{R}$ with the triangle inequality $d(x, y) + d(y, z) \geq d(x, z)$ and $d(x, x) = 0$. Actually the set of open balls is a base for the topology. The intersections and unions of balls are open too.

- the subspace (or relative) topology of a subset $S \subset T$, where T having the topology $\tau = \{O\}$ has as open sets $\tau_S = \{S \cap O\}$

- the product topology on the cartesian product set T1 $\times T2$is has as open sets $O1 \times O2$ where $O1$ en $O2$ are open sets in $T1$ and $T1$ respectively.

- the quotient topology of a space T having an equivalence relation $\equiv$ is the topology on the set T' of equivalence classes having as open sets the ones for which πX are open in T, where π is the natural projection from $T \to T'$. This topology is used when the equivalence relation consists of 'identifying' points (to make a line into a circle, a strip into a Möbius strip ect.).

- the open interval $(-1, 1)$ and the real line $\mathbb{R}$ are homeomorphic. The closed interval $[-1, 1]$ is not homeomorphic with the real line.

- A sphere $\{x^2 + y^2 + z^2 = r\}$ with one hole is homeomorphic to a closed disk. A sphere with two holes in it is homeomorphic to a cylinder. A sphere with one point missing is homeomorphic to the plane $\mathbb{R}^2$ (place the sphere on top of the plane and with the missing point as north pole. use this pole as start of projection lines to project all points of the sphere on the plane). A sphere missing two points is homeomorfic to an infinite long cylinder.

2.6.2 Homotopy

Besides the homeomorphic equivalence relation exists the homotopy equivalence relation. To define homotopy equivalence first the definition of homotopic functions is given. Two functions $f, g : T \to U$ are homotopic if there exists a homotopy, which is a continuous map $H : T \times [0, 1] \to U$, that joins f to g (for every $t \in T$ $H(t, 0) = f(t), H(t, 1) = g(t)$). Being homotopic is a equivalence relation on the set of continuous functions $f : T \to U$. A map is called null-homotopic if it homotopic to a constant function ($g : T \to \{c\}, c \in U$). Two topological spaces are homotopy equivalent if there exists maps $f : T \to U$ and $g : U \to T$, such that the compositions $fg : T \to T$ is homotopic to the identity on function T, id_T, and $gf : U \to U$ is homotopic to the identity function on U, id_U. Remember: for homeomorphisms one has two functions with $fg = id_T$ and $gf = id_U$. Being homeomorphic equivalent is a stronger condition than being homotopy equivalent.

An example of two homotopic functions is the identity function of a disk and a constant map from the disk to a constant point of the disk. The homotopy is the function $H((\phi, \rho), z) = ((1 - z)\rho, \phi)$, where $z \in [0, 1]$ and (ϕ, ρ) are polar coordinates. So, the identity map of a disk is a null-homotopic function. From this follows that a point and a disk are homotopy equivalent (use for f the constant function from the disk to the point and for g the function from the point to any point of the disk and fg will be homotopic to the identity on the disk and gf will be the identity function of the point)). A point and a euclidean space are also homotopy equivalent. A circle, a cylinder and an annulus are all homotopy equivalent.

3 Categories and morphisms

3.1 Definition category

A category is a set of objects and morphisms that satisfy a number of properties. You can talk for instance about the category of all sets, Set, where the objects are sets and the morphisms are functions. Or you can talk about the category of all toplogical spaces, Top, where the objects are topological spaces and the morphisms are the continuous functions. Category is treating all the categories on an equal footing and is not interested in the inner structure of the objects. A categorical statement is in terms of the objects and morphisms alone. As soon as you see something like $a \in A$ or $x_n \to x$ you are looking inside an object (a set resp. a topolocical space) and it is no longer a categorical statement. Category theory is interested in what can be said about the morphisms, regardless what the objects are. Well, you cannot say much perhaps, but what can be said applies to all those different categories!

A category $\mathcal{C}$ consists of:

1. a class of objects, $OBJ(C)$
2. for each $(a, b) : a \in OBJ(C), b \in OBJ(C)$ there is a set of morphisms $a \to b$, $Hom_C(a, b)$. If $f \in Hom_C(a, b)$ then a is called the domain of f and b the codomain of f
3. for each $(a, b, c) : a \in OBJ(C), b \in OBJ(C), c \in OBJ(C)$ there is a binary operation called composition $Hom_C(a, b) \times Hom_C(b, b) \to Hom_C(a, c)$
4. the composition is associative: $f(gh) = (fg)h$
5. every object $a \in OBJ(C)$ has an identity morphism $id_a \in Hom_C(a, a)$: $fid_a = f$ for each f with domain a and $id_a g = g$ for each g with codomain a
6. $Hom_C(a, b) \cap Hom_C(a', b') = \emptyset$ unless $a = a'b = b'$

That is a common definition. Some (insignificant) alternatives could be used. For instance when you think of computing languages there can be functions having the same name but working on different types, e.g. the function square sqr(x) having the same name for $sqr : \mathbb{R} \to \mathbb{R}$ and $sqr : \mathbb{N} \to \mathbb{N}$. If you think of these two functions as being the same then the last condition in the definition is not fulfilled. Some authors introduce the concept precategory to allow for the same functions to have different types. Another topic is the introduction of objects in the definition. If you have the identity morphisms then why still bother talking about the objects.

3.1.1 Examples of categories

- Set is the category with the class of all sets as objects and the functions as morphisms
- Top is the category with the class of all topological spaces as objects and the continuous functions as morphisms

- *Pos* is the category with the class of all partial ordered set as objects and the monotone functions as morphisms
- every poset $\mathcal{P}$ can be seen as a category with the underlying set as the objects and in every $Hom_{a,b}$ a morphism if $a \leq b$
- every monoid can be seen as a category with as only object the underlying set A and for every $a \in A$ $(a\cdot)(m)$ as morphisms $A \to A$
- *Grp* is the category with the class of all groups as objects and the group homomorphisms as morphisms.
- *Ab* is the category of abelian groups with the class of abelian groups as objects and the homomorphisms between abelian groups as morphisms.
- *Grph* is the category with the class of all graphs as objects and the graph homomorphisms as morphisms. Sometimes you will see that the injective homomorphisms are the morphisms of the category
- every directed graph can be seen as a category with the nodes of the graph as the objects and a morphism for every two nodes that are connected by a path
- **0** is the category with no object and with no morphism
- **1** is the category with one object x and with 1_x as the only morphism
- **2** is the category with two objects $x1, x2$ with a morphism $x1 \to x2$ and an identity morphism for both objects
- a discrete category is a category whose only morphisms are the identity morphisms

If $\mathcal{C}$ is a category, then one can define another category, the opposite (or dual) category $\mathcal{C}^{op}$, having the same objects as $\mathcal{C}$, but with the 'arrows of the morphisms' reversed: for every morphism f in $\mathcal{C}$ with domain a and codomain b there is a morphism f^{op} with domain b and codomain a. The composition of two morphisms in the dual category, $g^{op}f^{op}$, is the morphism $(fg)^{op}$. The dual of the dual categorie is the original category again: $(\mathcal{C}^{op})^{op} = \mathcal{C}$.

If $\mathcal{C}$ and $\mathcal{D}$ are two categories we can define a product category $\mathcal{C} \times \mathcal{D}$ where the objects are ordered pairs of objects (c, d) and the morfisms are ordered pairs (f_c, f_d) of morfisms. The composition of two morphisms (f_{c1}, f_{d1}) (f_{c2}, f_{d2}) equals $(f_{c1}f_{c1}, f_{d1}f_{d2})$.

If $\mathcal{C}$ is a category, and a is an object of $\mathcal{C}$, then one can define another category, the slice category $\mathcal{C}/a$, having as objects the morphisms of $\mathcal{C}$ with codomain a and as morphisms those morphisms f: $b \to c$ of $\mathcal{C}$ that one object of $\mathcal{C}/a$, o1: $b \to a$ is the composition of f with another object of $\mathcal{C}/a$, o2: $c \to a$, o2.f = o1. After the introduction of functors later on the categories with functors will be introduced. The slice category is a special form of a comma category.

3.2 Mono's, epi's, bijections and isomorphisms

Now we want to make categorical definitions. The first two are the definitions of a monomorphism (a mono) and of a epimorphism (an epi). A monomorphism is a concept related to an injective function in set theory and an epimorphism is a concept related to a surjective function. In the definitions of the only concepts that are used are category, object and morphism.

Let C be a category. A morfism f in C is called a monomorphism if $fg1 = fg2 \Rightarrow g1 = g2$ whenever domain(f) = codomain($g1$) = codomain($g2$) and domain($g1$) = domain($g2$).

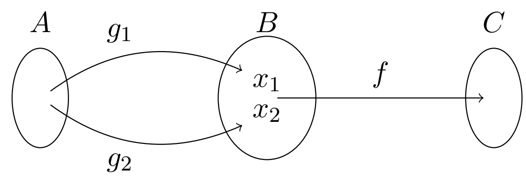


Figure 1: mono injective

In the category of sets a mono is an injective function. In figure 1 you see three sets A, B, C and two functions g1,g2 that have A as common domain and B as common codomain. Now suppose g1 and g2 map to two different elements $x1$ and $x2$ of B then the definition of a mono: $fg1 = fg2 \Rightarrow g1 = g2$ implies $f(x1) = f(x2) \Rightarrow x1 = x2$

A morfism f in C is called an epimorphism if $g1f = g2f \Rightarrow g1 = g2$ whenever codomain(f) = domain($g1$) = domain($g2$) and codomain($g1$) = codomain($g2$).

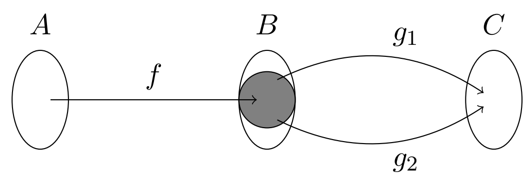


Figure 2: epi surjective

In the category of sets an epi is a surjective function. For imagine that f: $A \to B$ has an image that does not cover B, say only the gray area in figure 2 , then there are functions g1, g2: $B \to C$, that have the same mappings for the points in the gray area but a different value for some x not in the gray area. The we would have a situation where $g1f = g2f$ but not $g1 = g2$. This is not allowed, so the gray area and B must be the same.

For sets the statements are also true is the other direction: an injective function is a mono and a surjective function an epi.

A morphism f is defined to be a bijection when f is a mono and an epi. A morphism f: $a \to b$ is a isomorphism when f is invertable, i.e. there is a morphism g, an arrow in the other direction, with fg=1_b and gf=1_a. Every isomorphism is a bijection. In the category of sets a bijection is also a isomorphism. This is also an illustration that between two objects more than one ismorphisms can exist (for a set of n elements there are n! isomorphisms). It is possible that a morphism is a bijection but not a isomorphism (in the category **2** the morphism $x1 \to x2$ is an example (there is no inverse of this morphism).

A morphism f can have only one inverse. For, suppose it has two, g and h, then from $(gf)h = g(fh)$ it follows that $h = g$.

3.3 Duality

Another interesting definition, without mentioning the structure of the objects of a category, is that of a terminal object. A terminal object in $\mathcal{C}$ is an object A of $\mathcal{C}$ such that for every object B of $\mathcal{C}$ there is exactly one morphism $B \to A$. In Set terminal objects are one-element sets.

Let P be a property that some objects or morphisms have in a category. An example is the property that an object A is a terminal object, where the property is that for every object B of $\mathcal{C}$ there is exactly one morphism $B \to A$. Now lets look at the opposite category where the arrows of all morphisms are in the opposite direction. Then A is an object with the property that for for every object B of $\mathcal{C}^{op}$ there is exectly one morphism $A \to B$. This is the dual property. For every property stated in terms of a category $\mathcal{C}$ you can define a dual property in the opposite category. The dual property of being a terminal object has a name, it is the initial object. In the category of sets the initial object is the empty set and the morphisms the empty functions.

Note the use of the phrase 'exactly one morphism' (or 'unique morphism') in the definitions of terminal and initial objects. This phrase is used over and over agin in the definitions of new kinds of objects. A consequence of the use of this phrase is that, if you have two terminal (or initial) objects, say $T1$ and $T2$, then there exists a unique isomorphism between the two objects. So, not just an isomorphism, but a unique isomorphism. The proof in this case is straightforward.

Another example of dual properties is morphisms being mono or an epi. If a morphism of $\mathcal{C}$ has the property that it is a mono, $fg1 = fg2 \Rightarrow g1 = g2$, and you look at the morphisms in the opposite category $\mathcal{C}^{op}$, then that morphism has the property $g1f = g2f \Rightarrow g1 = g2$: it is an epi! The property of being an isomorphism is a self-dual property.

3.4 In other categories

We have seen what epi's, mono's terminals and initials are in the category of sets. And it is the first category one thinks of when defining new concepts. In other categories it may well be that objects or morphisms with these properties do not exist. In a 'strange' category where a particular directed graph is seen as a category, with the nodes of the graph as objects and a morphism for every two nodes that are connected by a path, every morphism is an epi and a mono and if there are terminal objects or initial objects depends on the graph.

As an example of a more common category let us look at *Grp*, the category of groups. This is an example of a category that has a set of elements in its objects. For this category it can be proved that a morphism g: $G1 \to G2$ is a mono (epi) if and only if the morphism g, seen as a set operation, is a mono (epi). For *Grp* initial and terminal objects are the same: it is the trivial group. Every group can be mapped to the trivial group T by mapping alle group elements to the single element of T. And the trivial group can be mapped to every group G by mapping its only element to the unit element of G. Objects that are initial and terminal at the same time are called zero objects. So, the trivial group is the zero object of *Grp*.

3.5 Other constructions

3.5.1 Products and coproducts

Besides terminal and initial obects it is possible to introduce many other interesting objects. The way it is done is always the same. In this section we see how it is done and will define an object that is the product and an other object that the coproduct of two objects A and B. After giving the definitions we will see what the definitions mean for a category like *Set*. In the text and in the figures we use the symbol U for the object that will be defined.

The first definition, product, uses morphisms in the direction from the object that is to be defined, U, towards the objects in terms of which the definition is made, A and B. In the definition of the coproduct the morphisms will be in the other direction.

A product of two objects A and B is defined as an object U together with two morphisms $\pi_a : U \to A$ and $\pi_b : U \to B$, called projection morphisms, having the property that for any other object V having projection morphisms f_a and f_b, there exists a unique morphism $m : V \to U$ such that $f_a = m\pi_a$ and $f_b = m\pi_b$.

Note that in this definition use is being made of the phrase 'for every V there exist a unique morphism'. It is the combination of 'there exists a morphism for any V we choose' and 'that morphism is unique' that can be used over and over again to prove the statements that we will make. The unique morphism is shown as a dashed line in the following diagram and has the name m.

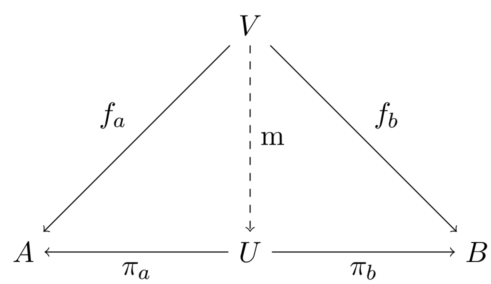


Figure 3: product

The defined object is called a product. Of course there is a relation between this definition and the cartesian product of sets in the category *Set* and products in other categories. In *Set* the cartesian product has as members elements (a, b) where $a \in A$ and $b \in B$. But in category theory the structure in the elements of the cartesian product can only expressed in terms of morphisms. This is done by introducing projection morphisms. A product is only a product in category theory if that product object has projection morphisms. There is no definition of a product in category theory without the projection morphisms.

But having projection morphisms is not enough. From the definition we know that all other sets V having two morphisms to A and B (let us cal these morphisms projection morphisms as well) will be projection morphisms that can be expressed using a unique morphism $m : v \to u$. What does this mean.

Define a function $\pi : U \to A \times B$ as $\pi(u) = (\pi_a(u), \pi_b(u))$. We will show now that this function is a bijection, so U is isomorphic with $A \times B$. Let (a, b) be an element of $A \times B$. There can always be found a set V with two morphisms f_a and f_b such that $f_a(v) = a$ and $f_b(v) = b$ for some $v \in V$ (you can choose even a one element set V). And following the definition there has to be a morphism m. This maps to a $u = m(v)$. And because $f_a = \pi_a m$ and $f_b = \pi_b m$ it follows that $\pi_a(u) = a$ and $\pi_b(u) = b$. So, for every (a,b) there exists a $u \in U$ with $(a, b) = \pi(u)$. So π is surjective and to prove this we used that for every V there exists a morphism m. From the uniqueness of m it follows that π is also injective. For let $(a, b) = \pi(u)$ be a point of $A \times B$ then there exists a set V with two morphisms f_a and f_b such that $f_a(v) = a$ and $f_b(v) = b$ for some $v \in V$ (again, you can choose even a one element set V). If there were another point u' with $(a, b) = \pi(u')$, then there are two morphisms $V \to U$ (the mapping of point $v \to u$ could be altered to a mapping $v \to u'$). But the m has to be unique. So π is also injective, and we have proved that π is a bijection between U and $A \times B$.

It does not mean that U and $A \times B$ are the same in *Set*. As far as category theory is concerned there can be more than one product and for instance in *Set* the two different cartesian products $A \times B$ and $B \times A$ can both be used as a product and they have the obvious unique isomorphism between them. The definition of a product in category theory comes close to the definition of a cartesian product in Set. How close? Up to an isomorphism.

As suspected the same concept of product can be used in the category Top (cartesian product with the product topology), Grp (direct product of groups) and many other categories. In the category of a poset $\mathcal{P}$ as a category of its own the product is the infimum (greatest lower bound or meet) of both objects.

Now we turn to the definition of the dual of the product, the coproduct. In this definition we use morphisms in the direction from the objects in terms of which the definition is made, towards the object that is to be defined, U as can be seen in the following figure.

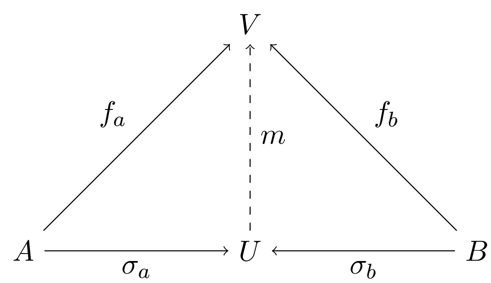


Figure 4: co-product

Instead of calling the morphisms σ_a and σ_b projections we now call them insertions.

A coproduct of two objects A and B is defined as an object U together with two morphisms $\sigma_a : A \to U$ and $\sigma_b : B \to U$, called insertion morphisms, having the property that for any other object V having insertion morphisms f_a and f_b, there exists a unique morphism $m : U \to V$ such that $f_a = m\sigma_a$ and $f_b = m\sigma_b$.

A coproduct is a combination of the U with the two insertion maps to A and B. Just as was the case with a product, there is no definition of a coproduct in category theory without the insertion morphisms. The defined object U is in some way 'the best' and this is expressed in the phrase 'such that whenever there is another object V with the morphisms f_a and f_b, then there is a unique morphism, $m : U \to V$, such that 'the diagram commutes', which means that $f_a = m\sigma_a$ and $f_b = m\sigma_b$.

In the category Set a coproduct is isomorphic to the disjoint union. To show this we define a new set $U_\sigma = \{\sigma_a(A)\} \cup \{\sigma_b(B\}$ and we see that the coproduct U is a subset of U_σ because else the mapping of the elements of $U \setminus U_\sigma \to V$ by m could take on any value, so m would not be unique. Because $U_\sigma \subset U$ by definition, we have $U_\sigma = U$. The functions σ_a and σ_b are injections. For suppose $\sigma_a(a1) = \sigma_a(a2)$, then we define a V with two functions such that $f(a1) \neq f(a2)$ and in that case there would exist no m that makes the diagram commute. The same reasoning shows that $\sigma_a(a)$ is alway different from $\sigma_b(b)$. Again we have used both parts of the phrase 'for every V there exists a morphism' and 'this morphism is unique' to prove that a coproduct and a disjoint union are really the same.

In the category of groups a coproduct is a free product of groups. In the category of a poset $\mathcal{P}$ as a category of its own the coproduct is the supremum (least upper bound or join) of both objects.

3.5.2 Equalizers and coequalizers

An equalizer is an object U together with a morphism q such that $fq = gq$ and such that for all other V with $hf = hg$ there is a unique morphism $m : V \to U$ such that $h = mq$. In the category Set an equalizer is isomorphic to the set $\{x : f(x) = g(x)\}$, where f, g are two functions $A \to B$. If you wander why in the following figure it is implicit that $fq = gq$ and $fh = gh$ then read the section on diagrams 4.3 and limits later on.

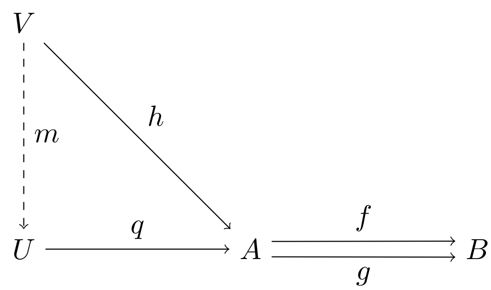


Figure 5: equalizer

A co-equalizer is an equalizer in the opposite category. It is left as an exercise to state the definition of a co-equalizer in words.

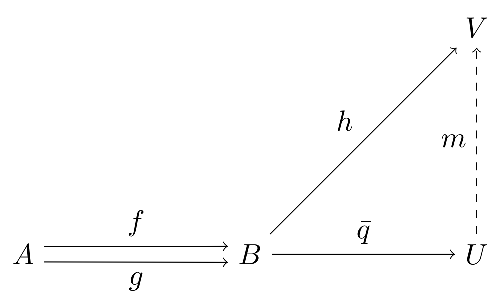


Figure 6: co-equalizer

In the figure it is shown that $hf = hg$ and $\bar{q}f = \bar{q}g$. For a co-equalizer in Set we can prove that $\bar{q}(B) = U$ in the same way as is done with the coproduct (otherwise m would not be unique). The coequalizer maps two elements from B to the same point $u \in U$ when there is one point $a1 \in A$ with $b1 = f(a1)$, $b2 = g(a1)$. There may be other points that are mapped to the same u. There may be another $a2$ that has for instance $b1 = f(a2)$, then $g(a2)$ is also mapped to the same u. The functions f and g do introduce a relation $R \subset B \times B$ and the equalizer function mappes the elements that are related to the same point in U. Given A and B with the functions f and g every h must do the same: every h mappes the elements that are related to the same point in V. But for it to be a co-equalizer it must be the smallest relation R on B. Or the smallest equivalence relation containing the smallest R (of course $\bar{q}(b1) = \bar{q}(b1)$ and if $\bar{q}(b1) = \bar{q}(b2), \bar{q}(b2) = \bar{q}(b3) \Rightarrow \bar{q}(b1) = \bar{q}(b3)$, so for the equalizer function we are interested in equivalence realtions on B). It has to be the smallest equivalence relation so that U has the greatest number of elements. This is necessary for $m : U \to V$ to exist.

3.5.3 Pullbacks and pushouts

The definition of a pullback of g along f (or a pullback of f along g) is given using the following figure

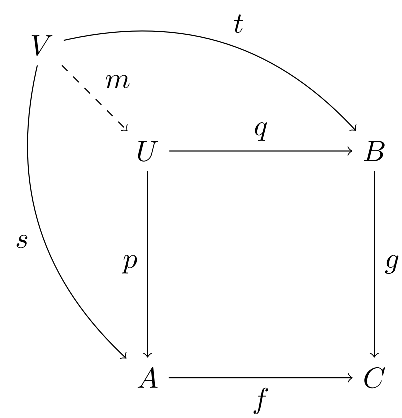


Figure 7: pullback

In the centre of the definition is a commuting square from U, this means with morphisms $fp = gq$. So if ones sees U as a product of $A \times B$, then one must be aware of the restriction put on this product by the morphisms f and g: the square must commute. And to be a pullback one has to find the best possible commuting square. If there is another V with projections s and t that makes a commuting square, then there has to be a unique morphism m, such that $s = mp$ and $t = mq$.

In the category of *Set* the pullback is just the set $\{(a, b) : f(a) = g(b)\}$.

A pushout is the dual to a pullback and is given by the following figure

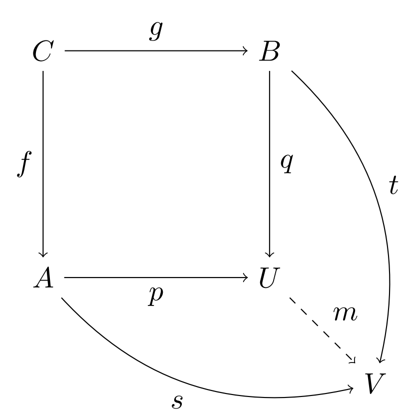


Figure 8: pushout

The pullback is a modified coproduct (disjoint union), the modification given by the commuting square ($qg = pf$). In the categoryy *Set* the points in A and B are mapped to a common $u \in U$ when they are mapped from a common point in C by f and g. All the mappings t and

s to V do this to their V's. But the pushout is the one with a minimum of points of A and B identified.

An interesting example of a pushout occurs when one of the mappings f or g, lets assume f, is the inclusion map, C being a subspace of A. In that case every element c of A is identified with the element $g(c)$ of B. We then have a gluing of B and A over the subset C.

4 Functors

4.1 Definition functor

Now we know that there are (many) categories it is natural to think about mappings from one category to another. Of course we think of some structure preserving morphism, one that takes objects to objects, arrows to arrows, identities to identities, compositions to compositions. Then we have something called a functor.

Let $\mathcal{C}_{\mathcal{A}}$ and $\mathcal{C}_{\mathcal{B}}$ be categories. A (covariant) functor F: $\mathcal{C}_{\mathcal{A}} \to \mathcal{C}_{\mathcal{B}}$ is a function that associates with each object a in $\mathcal{C}_{\mathcal{A}}$ an object $F(a)$ in $\mathcal{C}_{\mathcal{B}}$ and to each morphism $f : a \to b$ in $\mathcal{C}_{\mathcal{A}}$ a morphism $F(f) : F(a) \to F(b)$ in $\mathcal{C}_{\mathcal{B}}$ such that

1. $F(gf) = F(g)F(f)$ whenever gf is defined
2. $F(id_a) = id_{F(a)}$ for each a

A contravariant functor F is a covariant functor from the opposite category $\mathcal{C}^{op} \to \mathcal{C}$ having $F(gf) = F(f)F(g)$ whenever gf is defined.

4.1.1 Examples of functors

- constant functor. A functor F: $\mathcal{C}_{\mathcal{A}} \to \mathcal{C}_{\mathcal{B}}$ that maps every object of $\mathcal{C}_{\mathcal{A}}$ to a single object X of $\mathcal{C}_{\mathcal{B}}$ and every morphism to the identity morphism of X.
- forgetful functor. A functor F: $\mathcal{C}_{\mathcal{A}} \to \mathcal{C}_{\mathcal{B}}$ from a setoriented category $\mathcal{C}_{\mathcal{A}}$ (a category where the objects are sets, for instance Top, Grp) to the category Set, where all objects are mapped to the underlying set and the homomorphisms or homeomorphisms are mapped to the underlying functions.

4.2 Faithfull and full functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $F : \mathcal{C} \to \mathcal{D}$ be a functor. For every pair of objects X and Y in $\mathcal{C}$ we have a set of morphisms $Hom_{\mathcal{C}}(X, Y)$. A functor F is called faithful if for every such set $Hom_{X,Y}$ the induced function $f : Hom_{\mathcal{C}}(X, Y) \to Hom_{\mathcal{D}}(F(X), F(Y))$ is injective. A functor F is called full if the induced function $f : Hom_{\mathcal{C}}(X, Y) \to Hom_{\mathcal{D}}(F(X), F(Y))$ is surjective.

An example is the inclusion map $i : Ab \to Grp$ where this map is faithful as well as full (also called fully faithful).

4.2.1 Cat

Given two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ one can define the composition $GF : \mathcal{C} \to \mathcal{E}$. This composition is associative and since any category $\mathcal{C}$ has the identity functor $\mathcal{C} \to \mathcal{C}$ we have a new category Cat which has categories as objects and functors as morphisms.

4.2.2 Comma category

Let S and T be functors to the category $\mathcal{C}$, $S : \mathcal{A} \to \mathcal{C}$ and $T : \mathcal{B} \to \mathcal{C}$. The objects of the comma category $S \downarrow T$ are triples $(a, b, f) : a \in \mathcal{A}, b \in \mathcal{B}, f \in Hom_C$. The morphisms $(a_1, b_1, f_1) \to (a_2, b_2, f_2)$ are pairs $(g, h) : g \in Hom_A, h \in Hom_B$, such that the following diagram commutes

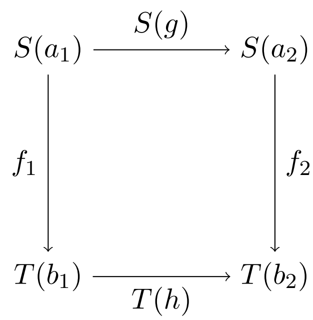


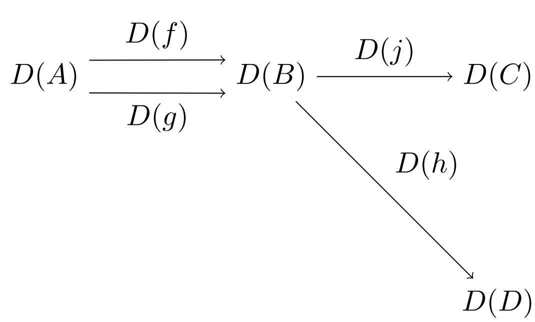
Figure 9: comma category

4.3 Diagrams

Now we know what a functor is we can give a formal definition of what a diagram is. And in the next section we will look back at the constructions we made before (of terminal object, product, equalizer etc.) and see how we can define all this using limits and co-limits.

A diagram is as the name suggests a selection of objects and morphisms from a category that you are drawing in a figure. It are the objects and morphisms you are interested in. Of course in mathematical speech this can be made more formal without thinking of a figure and this is done introducing an index category $\mathcal{J}$, somewhat similar to introducing an index when talking of subsets A_i of a set.

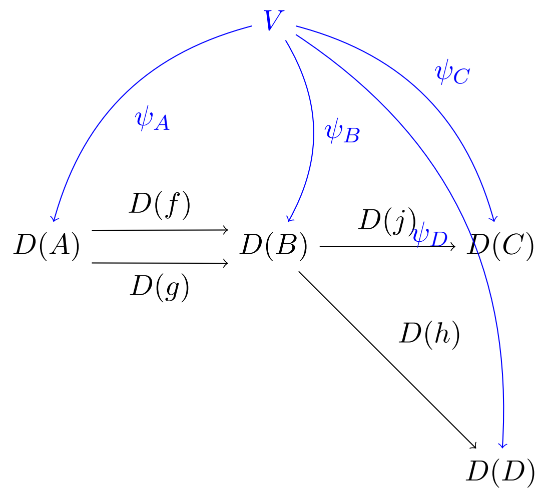
A diagram in a category $\mathcal{C}$ with this index category $\mathcal{J}$ is then a functor $D : \mathcal{J} \to \mathcal{C}$. Often one is interested in index categories where $\mathcal{J}$ is finite or even discrete (discrete: the only morphisms in $\mathcal{J}$ are the identity morphisms). The following figure shows objects in $\mathcal{C}$ that are maped by D where there are four objects in $J(A, B, C, D)$ and also four morphisms in $J(f, g, h, j)$.



4.4 Limits and colimits

A limit of a diagram is a universal cone over the diagram. So first we will see what a cone is and then what makes it a universal cone.

A cone is an element of a category $\mathcal{C}$ together with a set of morphisms to a diagram that 'makes all triangles commute'. In the figure we show the same diagram again, now together with the vertex of the cone, V and morphisms ψ of the cone.



The cone is said to be from V to D. The morphisms go from V i the direction of the diagram. To be a cone all triangles must commute, e.g. $\psi_B = D(f)\psi_A = D(g)\psi_A$. The dual notion of a cone is a co-cone with all arrows ψ reversed. The the cone is then said to be from the diagram D to the vertex V.

A limit is a universal cone over a diagram. As usual the definition is made by introducing another cone (think of another cone as copying everything in blue to another color with a vertex V and morphisms ϕ. The cone U with ψ is universal if for every such 'colored copy' there is a unique morphism m from U to V such that $\phi = m\psi$ for all the objects in the diagram.

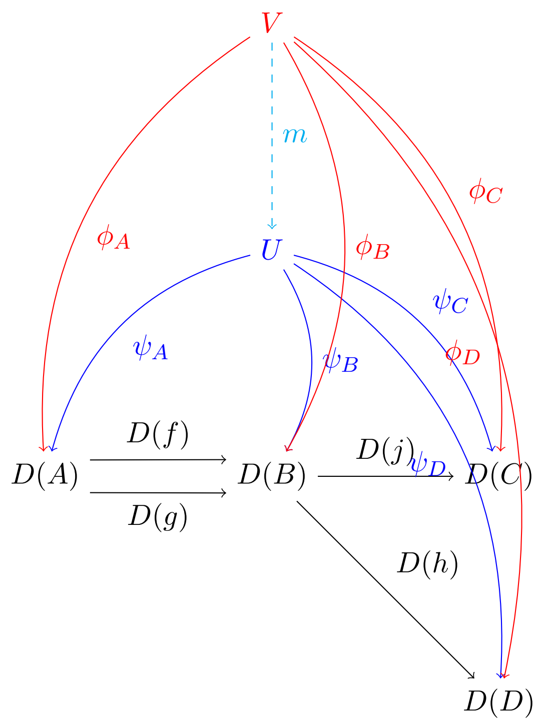


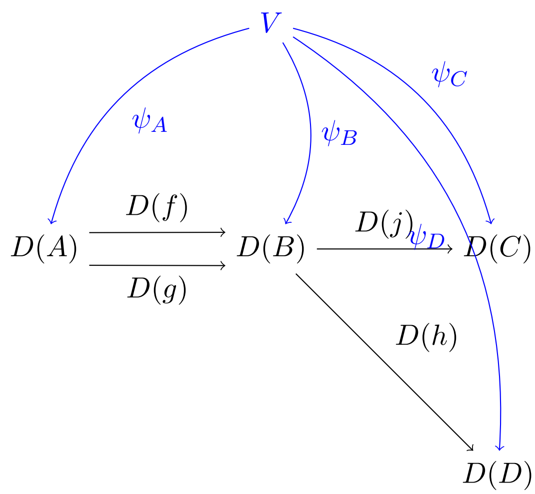
Figure 10: limit

A co-limit is a limit in the opposite category. A co-limit uses a universal co-diagram in the definition with a unique morphism going from the 'universal vertex' U to V.

4.4.1 Examples limits and colimits

All universal constructions we met so far can be redefined using limits and colimits. When the diagram is an empty diagram then the limit is a terminal object and the colimit is a initial object. When the diagram has two objects and no morphisms then the limit is the product and the colimit is the co-product. etc.

Let us look again at our example diagram and see what it means that we have 'commutative triangles' in our definition of a cone (or co-cone).



Commutative triangles here give us:

1. $\psi_B = D(f)\psi_A$
2. $\psi_B = D(g)\psi_A$
3. $\psi_C = D(j)\psi_B$
4. $\psi_D = D(h)\psi_B$

The first two equations give us $D(f)\psi_A = D(g)\psi_A$. And the last two equations can also be written as $\psi_C = D(j)D(f)\psi_A$ and $\psi_D = D(h)D(f)\psi_A$. The morphism ψ_B that points to the intermediate node $D(B)$ is not present in these equations. If you look at the figure used in the definition of an equializer (fig 5) you will see that this morphism from V to B (and from U to B) is not drawn. And when the figure is made for the definition of the pullback (fig. 7) you see that the morphism from A to C is not drawn either.

4.5 Natural transformations

4.5.1 Natural transformations

A natural transformation morphs one functor to another. Not every morph between two functors is a natural transformation, of course. So, there is an extra 'naturality' condition.

Suppose we have two categories, $\mathcal{C}$ and $\mathcal{D}$ and we have two functors $F, G : \mathcal{C} \to \mathcal{D}$. For every X in $\mathcal{C}$ we have an object $F(X)$ and $G(X)$ in $\mathcal{D}$. For every X in $\mathcal{C}$ all the morphisms in $\mathcal{C}, m_{\mathcal{C}} : X \to Y$ are mapped by F and G to morphisms $m_{\mathcal{D}} : F(X) \to F(Y)$ in $\mathcal{D}$. Then a natural transformation is a collection of morphisms, one for every $X \in (C)$, α_X in $\mathcal{D}$, that morphs $F(X) \to G(X)$, having the naturality condition that for every $m_{\mathcal{C}}$ the following square in $\mathcal{D}$ commutes:

$$
\begin{array}{ccc}
F(X) & \xrightarrow{F(m_{\mathcal{C}})} & F(Y) \\
\Big\downarrow \alpha_X & & \Big\downarrow \alpha_Y \\
G(X) & \xrightarrow{G(m_{\mathcal{C}})} & G(Y)
\end{array}
$$

Figure 11: naturality

It is important that a natural transformation has for every X a morphism in $\mathcal{D}$, $m_{\mathcal{D}}$, otherwise the natural transformation can not be constructed, and that these morphisms make the square in $\mathcal{D}$ commute for all morphisms $\mathcal{C}, m_{\mathcal{C}} : X \to Y$.

4.5.2 Examples of natural transformations

- Let $\mathcal{C}$ and $\mathcal{D}$ be two partially ordered sets, both seen as a category in themselves: the objects of the two categories are the members of the set and the morphisms of the two categories indicate if elements of the set have the leq relation. A functor $\mathcal{C} \to \mathcal{D}$ is a monotone function. There is a unique natural transformation between two functors F and G if $F(X) \leq G(X)$ for every $X \in \mathcal{C}$. If there is a $X \in \mathcal{C}$ for which $F(X) \not\leq G(X)$ then we have the situation where for this X there is no possible candidate for a morphism $m_{\mathcal{D}}$ in $\mathcal{D}$, so the natural transformation can not be constructed.

- Let $\mathcal{C}$ and $\mathcal{D}$ both be the category Set. Then there is no natural transformation from the trivial functor (the constant functor that maps every object X in $\mathcal{C}$ to a one-element set in $\mathcal{D}$ and every morphism in $\mathcal{C}$ to the identity morphism of this element) to the identity functor. For a proof look at morphisms (in Set functions) $f : X \to Y$ and let X be the one-element set, that we will call T (for terminal). Now look at the figure of the commuting square in figure 11 for this situation: $F(X)$, $F(Y)$ and $G(X)$ are all equal to T. $F(m_C)$ is the identity function. If Y has more than one element then there are more than one different functions f. If the functions α_X and α_Y are chosen then the square has to commute for all the possible functions f but if it does for one it will not do for another f.

- we defined a homotopy before, see 2.6.2. If we have topological spaces T and U and functions $f, g : T \to U$, then f and g are homotopic if there exists a homotopy, a continuous function $H : T \times [0, 1] \to U$, that joins f to g. We can think of this as a collection of paths in U indexed by T. So, for each $t \in T$, there is a path $H(t, \sigma)$ from $f(t)$ to $g(x)$. We can transform this definition into a natural transformation by introducing T and U as two categories with the points of the spaces as objects and with a morphism between two points for each homotopy class of paths between the points. The relation of being homotopic is an equivalence relation on the paths between two fixed points in a topological space.

4.5.3 Functor category

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. We can form the category of functors from $\mathcal{C}$ to $\mathcal{D}$, $\mathcal{C}^{\mathcal{D}}$, that has the functors from $\mathcal{C} \to \mathcal{D}$ as objects and has as morphisms the natural transformations between such functors. Natural transformations can be composed: if $\mu(X) : F(X) \to G(X)$ is a natural transformation from the functor $F : \mathcal{C} \to \mathcal{D}$ to the functor $G : \mathcal{C} \to \mathcal{D}$ and and $\nu(X) : G(X) \to H(X)$ is a natural transformation from the functor $G : \mathcal{C} \to \mathcal{D}$ to the functor $H : \mathcal{C} \to \mathcal{D}$, then the collection $\nu(X)\mu(X) : F(X) \to H(X)$ defines a natural transformation from F to H. With this composition $\mathcal{C}^{\mathcal{D}}$ satisfies the axioms of a category.

4.5.4 Adjoint functor

... see catsmoeder... see Saunders Mac Lane ...

5 More

5.1 More on the internet

- an introduction to category theory in four easy movements. By A Schalk and H Simmons. A book that has the same subjects as this paper. Some 200 pages, with exercises.

  http://www.cs.man.ac.uk/ hsimmons/BOOKS/CatTheory.pdf

- a chapter that covers the same subjects but only 7 pages long. It is a chapter from a book that covers many topics from algebra.

  http://www.math.northwestern.edu/ len/d70/chap11.pdf

- Basic category theory, Jaap van Oosten, juli 2002. The first three chapters cover the same as this paper but a little more in depth. After that a number of different subjects follow.

  http://www.staff.science.uu.nl/ ooste110/syllabi/catsmoeder.pdf

- category theory for computing science, Michael Barr and Charles Wells. A second book on category theory, this one tuned to students in IT. It does talk about the functional programming languages as categories and later on in Chapter 6 about lambda calculus. Remark: there is anothe book by the same authors on the web (toposes, triples and theories, Michael Barr and Charles Wells, tr12.pdf) that is much more advanced.

  www.tac.mta.ca/tac/reprints/articles/22/tr22.pdf

- a large number of videos on YouTube by the Catsters.

  http://www.youtube.com/user/TheCatsters?feature=watch

- an internet page with more links to books and articles on category theory

  http://dekudekuplex.wordpress.com/2009/01/16/learning-haskell-through-category-theory-and-adventuring-in-category-land-like-flatterland-only-about-categories/

5.2 More in books

- Categories for the working mathematician, Saunders Mac Lane (1909-2005), ISBN: 0-387-98403-8, second rev. edition 1998, 314 pages